MINERVA INVESTMENT MANAGEMENT SOCIETY

Quantitative Research Division

Options pricing with Monte Carlo simulations

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Options introduction

Options are derivative instruments whose value is determined by the value of one or more securities known as underlings. These objects are traded both on exchanges and in the over-the-counter market. Puts and calls are the two basic forms of options and for this reason they are called *Plain Vanilla* options. The owner of a call option has the right, but not the obligation, to purchase an asset within a certain given time period and at a specified price. The put options provide the owner the right, but not the responsibility, to sell an asset within a certain time frame and at a specified price. The price in the contract is known as the *strike price* (K), and the date of expiry is known as *maturity* (T). The European options can only be exercised at maturity, but the American options can be exercised at any time before maturity. It should be stressed that an option grants the holder the power to make decisions. This privilege is not required to be exercised by the possessor. This differentiates options from forwards and futures contracts, in which the holder is required to acquire or sell the underlying asset. Since you have the right to engage into the contract, the payoff at maturity of a call or put are always positive, you must pay an initial fee known as the *premium* or *price*. Of course, in the contract there are two counterparts: a buyer and a seller, known as a *writer* of the option.

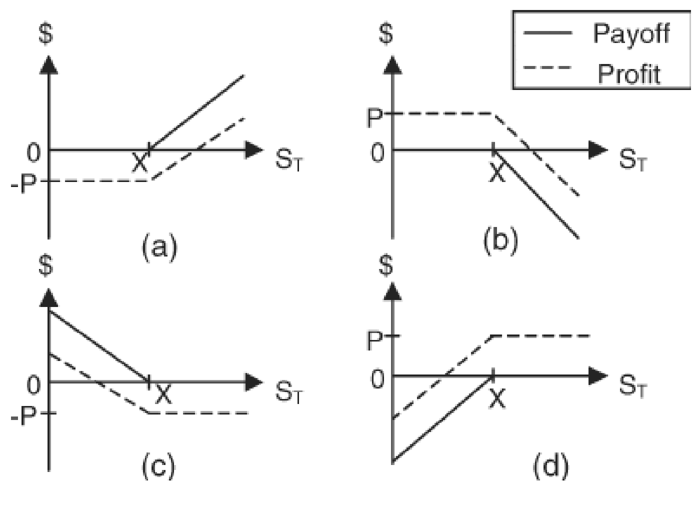

Figure 1: Payoff and P/L of calls and puts

The relative payoffs at maturity are opposite to each other and the formulas are the followings:

$$\begin{aligned}
&(a) \quad \text{Payoff Call buyer} = \text{Max}(S_T - K, 0)\\
&(b) \quad \text{Payoff Call seller} = \text{Min}(K - S_T, 0)\\
&(c) \quad \text{Payoff Call buyer} = \text{Max}(K - S_T, 0)\\
&(d) \quad \text{Payoff Call seller} = \text{Min}(S_T - K, 0)
\end{aligned}$$

You can see the graphical representations at the **Figure 1**.

Options Pricing with Monte Carlo simulations

Option pricing theory calculates the value of an options contract by allocating a premium based on the projected chance that the contract will expire in the money (ITM). Option pricing theory, in essence, gives an assessment of an option's fair value, which traders integrate into their strategy. To hypothetically value an option, pricing models take into account variables such as current market price, strike price, volatility, interest rate, and time to expiry. Black-Scholes, binomial option pricing, and Monte Carlo simulation are some popular models for valuing options. When analytic solutions such as the Black-Scholes model, are not available, numerical approaches for valuing derivatives are helpful. Monte Carlo simulation is typically employed for derivatives when the payoff is dependent on the underlying variable's history or if there are several underlying variables. It employs the risk-neutral valuation conclusion when valuing an option. We sample pathways to determine the anticipated payout in a risk-free environment, and then discount this payoff at the risk-free rate. Consider a derivative with a payment at time T that is dependent on a single market variable S. Assuming constant interest rates, we can calculate the derivative as follows:

1. In a risk-free universe, choose a random path for S.

2. Determine the payment based on the derivative.

3. Repeat steps 1 and 2 to get a large number of sample values of the derivative's reward in a risk-free environment.
4. Estimate the anticipated payout in a risk-free world by taking the mean of the sample payoffs.
5. Discount this expected payoff at the risk-free rate to get an estimate of the value of the derivative.

Suppose that the process followed by the underlying market variable in a risk-neutral world is

$$dS = \hat{\mu} S dt + \sigma S dz \tag{1}$$

where dz is a Wiener process, $\hat{\mu}$ is the expected return in a risk-neutral world, and σ is the volatility. To simulate the path followed by S, we can divide the life of the derivative into N short intervals of length δt and approximate equation (1) as

$$S(t + \Delta t) - S(t) = \hat{\mu} S(t) \Delta t + \sigma S(t) \epsilon \sqrt{\Delta t} \tag{2}$$

where $S(t)$ denotes the value of S at time t, ϵ is a random sample from a normal distribution with mean zero and standard deviation of 1.0. This allows you to compute the value of S at time Δt from the initial value of S, the value at time $2\Delta t$ from the value at time Δt, and so on. In practice, it is usually more accurate to simulate ln S rather than S. From Itô's lemma the process followed by ln S is

$$d \ln S = \left(\hat{\mu} - \frac{\sigma^2}{2} \right) dt + \sigma dz \tag{3}$$

so that

$$S(t + \Delta t) = S(t) \exp \left[\left(\hat{\mu} - \frac{\sigma^2}{2} \right) \Delta t + \sigma \epsilon \sqrt{\Delta t} \right] \tag{4}$$

This equation is used to construct a path for S. Also, if $\hat{\mu}$ and σ are constant, then

$$S(T) = S(0) \exp \left[\left(\hat{\mu} - \frac{\sigma^2}{2} \right) T + \sigma \epsilon \sqrt{\Delta t} \right] \tag{5}$$

This equation can be used to value derivatives that provide a non-standard payoff at time T. The main benefit of Monte Carlo simulation is that it may be utilized when the payout depends on both the path taken by the underlying variable S and the ultimate value of S. Payoffs might occur at various points over the life of the derivative rather than all at once. For S, any stochastic process may be accommodated. The approach may also be modified to cover instances when the payoff from the derivative is dependent on numerous underlying market factors. The disadvantages of Monte Carlo simulation are that it takes a long time to compute and cannot readily handle scenarios when there are early exercise opportunities.

Asian Options

Asian options are path dependent exotic derivatives, whose payoff does not depend only on the terminal price of the underlying asset, but also on the price path during the life of the option. Being more precise, 3 In an Asian option the final payoff depends on the average price of the underlying asset, observed during the life of the option at some precise dates, called fixings.

Because of this fact, Asian options have a lower volatility and hence rendering them cheaper relative to their European counterparts. They are commonly traded on currencies and commodity products which have low trading volumes, and they are primarily employed for corporate hedging. There are four main types of Asian options: Continuous arithmetic average Asian call or put

$$\Phi(S) = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+ \text{ or } \Phi(S) = \left(K - \frac{1}{T}\int_0^T S(t)dt\right)^+$$

Continuous geometric average Asian call or put

$$\Phi(S) = \left(e^{\frac{1}{T}\int_0^T \log S(t)t} - K\right)^+ \text{ or } \Phi(S) = \left(K - e^{\frac{1}{T}\int_0^T \log S(t)dt}\right)^+$$

Discrete arithmetic average Asian call or put

$$\Phi(S) = \left(\frac{1}{m+1}\sum_{i=0}^{m} S\left(\frac{iT}{m}\right) - K\right)^+ \text{ or } \Phi(S) = \left(K - \frac{1}{m+1}\sum_{i=0}^{m} S\left(\frac{iT}{m}\right)\right)^+$$

Discrete geometric average Asian call or put

$$\Phi(S) = \left(e^{\frac{1}{m+1}\sum_{i=0}^{\pi} \log s\left(\frac{i}{m}\right)} - K\right)^+ \text{ or } \Phi(S) = \left(K - e^{\frac{1}{m+1}\sum_{\sum=0}^{\infty} \log s\left(\frac{i}{m}\right)}\right)^+$$

Pricing a geometric average Asian option

By assuming that the underlying asset's price follows a log-normal distribution continuous in time, we know that the product of log-normal distributed random variables is also log-normal distributed, while the same does not apply to the sum. Consequently, that the pricing of geometric average Asian options should be easy to deal with, since its formula can be derived in the Black-Scholes framework. On the contrary, arithmetic average options are more complicated to handle, and must be priced used simulation approaches.

Starting from a geometric Asian call, we can derive its price through the risk neutral method:

$$\begin{aligned} C^{K,g}\left(S_0,T\right) &= \exp(-rT)E\left(\left(\prod_{i=0}^{m} S\left(\frac{iT}{m}\right)\right)^{1/(m+1)} - K\right)^{+} \\ &= \exp((\rho-r)T)\exp(-\rho T)E\left(S_0\exp\left(\left(\rho-\sigma_Z^2/2\right)T+\sigma_Z\sqrt{T}Z\right)-K\right)^{+} \\ &= \exp((\rho-r)T)C^{K}\left(S_0,T\right) \end{aligned}$$

Where

$$\sigma_Z = \sigma\sqrt{\frac{2m+1}{6(m+1)}}$$
$$\rho = \frac{(r-\sigma^2/2)+\sigma_Z^2}{2}$$

And $C^k\left(S_0,T\right)$ denotes the price of a European call option with risk-free interest rate ρ and the volatility σZ.

We can clearly see that this formula, albeit with some differences, resembles the plain European pricing formula. In addition, from this formula we can derive the standard Black-Sholes Formula.:

$$\begin{aligned} C^{K,g}\left(S_0,T\right) &= \exp((\rho-r)T)C^{K}\left(S_0,T\right) \\ &= \exp((\rho-r)T)\left(S_0\Phi\left(\tilde{d}_1\right)-K\exp(-\rho T)\Phi\left(\tilde{d}_2\right)\right) \\ &= \exp(-rT)\left(S_0\exp(\rho T)\Phi\left(\tilde{d}_1\right)-K\Phi\left(\tilde{d}_2\right)\right) \end{aligned}$$

Where d_1 and d_2 are computed based on the European call option with risk-free interest rate ρ and the volatility σZ. The same reasoning can be applied to put options, arriving at the following formula:

$$\begin{aligned} P^{K,g}(S_0,T) &= \exp((\rho-r)T)\overline{P^K}(S_0,T) \\ &= \exp((\rho-r)T)\left(K\exp(-\rho T)\Phi\left(-\tilde{d}_2\right)-S_0\Phi\left(-\tilde{d}_1\right)\right) \\ &= \exp(-rT)\left(K\Phi\left(-\bar{d}_2\right)-S_0\exp(\rho T)\Phi\left(-\bar{d}_1\right)\right) \end{aligned}$$

Arithmetic Asian option Monte Carlo pricing

As we previously mentioned, there is no close formula to properly price arithmetic Asian options. Therefore, our best alternative is to estimate it by employing Monte Carlo simulation. This section will briefly go through the Monte Carlo framework and then will focus on our implementation. Suppose we want to estimate some value θ, and we have

$$\theta = E(g(X))$$

where $g(x)$ is an arbitrary function such that $E(|g(X)|) < \infty$, then we could generate n independent random observations $X1, X2, \ldots, Xn$ from the probability function $f(X)$. The estimator for θ is given by

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} g(X_i)$$

Since $E(|g(X)|) < \infty$, we can get by strong law of large number

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} g(X_i) \stackrel{a.s.}{\to} Eg(X) \text{ as } n \to \infty \\ &\hat{\theta} \to \theta \quad \text{ as } n \to \infty. \end{aligned}$$

The Monte Carlo simulation is never exact, and one always has to take the sample variance into account. It can be expressed as

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(g(X_i)-\hat{\theta}\right)^2$$

The central limit theorem tells us that

$$\sqrt{n}\frac{(\hat{\theta}-\theta)}{s} \rightarrow N(0,1) \text{ as } n \rightarrow \infty$$

Against this background, we can obtain a solid estimate of the option price, and the associated confidence interval, by simulating a sufficiently large number of price paths for the underlying asset. However, the crude Monte Carlo method is very inefficient because of its slow convergence rate. This is because the standard deviation of the estimate is inversely proportional to the square root of n. As a result, the model requires an extremely high number of simulations only to reach a sufficient degree of accuracy. We can solve this problem by implementing variance-reduction techniques. Here we will discuss two of the most important ones.

The first variance reduction methodology is antithetic variate method, which takes advantage of the fact that the variance of the sum of two negatively correlated variables is lower than the sum of the variances. As a consequence, the new estimation formula is defined as follows

$$\bar{\theta}_{\alpha} = \frac{1}{n}\sum_{i=1}^{n}\frac{g(X_i)+g(-X_i)}{2}, \quad \text{with i.i.d. } X_i \sim N(0,1),$$

Where $-X$ is the antithetic variate of X. In our case, we used the opposite of the simulated price path as the antithetic variable. It must be noted that this technique can be applied only if $g(X)$ is monotonic. The second technique is the control variate method. This methodology is based on the concept of including in the estimator an additional variable Z with known mean E(Z). The new estimator and its expected value are defined as

$$\hat{\theta}_c = Y + c(Z - E(Z))$$
$$E\left(\hat{\theta}_c\right) = E(Y) + c(E(Z) - E(Z)) = E(Y) = \theta$$

By implementing the methodology, we can considerably lower the variance of the estimate as long as the two variables are correlated

$$\begin{aligned}\operatorname{Var}\left(\hat{\theta}_{c_{\text{me}}}\right) &= \operatorname{Var}(Y) + {c_{\min}}^2 \operatorname{Var}(Z) + 2c_{\min} \operatorname{Cov}(Y, Z) \\ &= \operatorname{Var}(Y) - \frac{\operatorname{Cov}(Y, Z)^2}{\operatorname{Var}(Z)} \\ &= \operatorname{Var}(\bar{\theta}) - \frac{\operatorname{Cov}(Y, Z)^2}{\operatorname{Var}(Z)}.\end{aligned}$$

In our analysis, we used the value of the geometric Asian option with the same characteristics as the arithmetic one. Its price is known through the closed Black-Scholes formula and is positively correlated with the price of the arithmetic one, so it is the ideal candidate for this role.

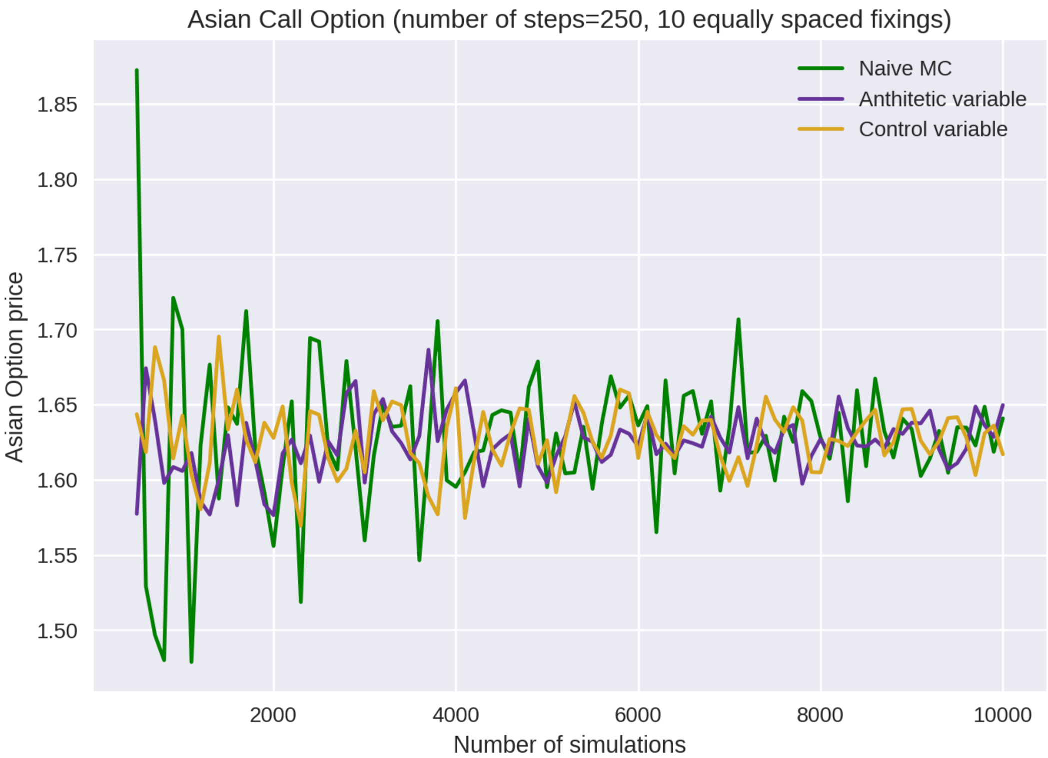

Figure 2: Convergence of Monte Carlo Estimator

The following table shows the convergence of the estimated price as the number of simulations increases. As expected, the crude MC method displays the highest variance, even if it subsides by increasing the number of simulations. The antithetic and control variate methods exhibit similar performance. By a number of simulations in the ballpark of 10000, all of the three methods converge to approximately the same value. However, it must be noted that these techniques cannot always be implemented: the antithetic variate requires a monotonic function for the payoffs, while the control variable cannot be properly identified for each problem.

Barrier Options

Barrier options are another type of path-dependent option. As the name suggests, the payoff of the option is based on whether the price S of the underlying asset has crossed a set level, i.e., if it has passed the set barrier H. Barrier options are traded in the OTC derivatives market and attracts participants due to its lower price (in comparison with the corresponding regular options). The barrier options can be divided into "in" and "out" options based on the implications of the crossing of the barrier threshold. In knock-in options the contract is valid if the price crosses the barrier. On the other hand, in knock-out options it is cancelled if the barrier is crossed. To sum up, the option pays at maturity if the barrier condition (in or out depending on the specific option) holds and if the option is in the money.

Another characteristic of the barrier options is whether the option is "up" or "down". This characteristic of the option is defined by the relationship between the current asset price S_0 and the set barrier H. If $H > S_0$, then the option is an up option, otherwise if $H < S_0$ then it is a down option.

Table 1 presents a short review of the main types of barrier options discussed above:

An interesting property of this exotic option is the so called knock in/knock out parity. According to the parity, the combination of an "in" and an "out" barrier option yields the same value as the corresponding vanilla option, i.e.,

$$C = C_{in} + C_{out} \text{ for call options}$$

Name	Contract Validity Condition
Up-and-in	S must move upwards to cross the barrier and validate the contract
Down-and-in	S must move downwards to cross the barrier and validate the contract
Up-and-out	If S moves upwards and crosses the barrier the contract is invalid
Down-and-out	If S moves downwards and crosses the barrier the contract is invalid

Table 1: Main types of barrier options

and

$$P = P_{in} + P_{out} \text{ for put options.}$$

The barrier condition can be monitored in a discrete or continuous manner. The frequency of the observation of the price of the underlying asset S raises an important issue when considering barrier options. The monitoring of the price is crucial as it allows us to understand if the barrier has been crossed and thus, if the contract is still valid or not. The price of a barrier option can be found using analytical formulas if we assume that we monitor S continuously. In particular, if $H \leq K$ (the barrier is less or equal to the strike price) the value of a down-and-in call at time 0 is

$$c_{di} = S_0 e^{-qT}(H/S_0)^{2\lambda} N(y) - Ke^{-r}(H/S_0)^{2\lambda-2} N(y - \sigma\sqrt{T})$$

where

$$\lambda = \frac{r-q+\sigma^2/2}{\sigma^2}$$

$$y = \frac{ln(H^2/S_0K))}{\sigma\sqrt{T}}$$

We can find easily the value of the respective down-and-out call option by considering the knock-in/knock-out parity we discussed before. Hence, the value of the call is:

$$c_{do} = c - c_{di}$$

In case the barrier is higher than the strike price ($H > K$) the value of the down-and out- call option is:

$$c_{do} = S_0N(x1)e^{-qT} - Ke^{-rT}N(x_1 - \sigma\sqrt{T}) - S_0e^{-qT}(H/S_0)^{2\lambda}N(y_1) + Ke^{-rT}(H/S_0)^{2\lambda-2}N(y_1 - \sigma\sqrt{T})$$

where

$$x_1 = \frac{ln(S_0/H)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

$$y_1 = \frac{ln(H/S_0}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T}$$

In case $H > K$ the price of the option is given by:

$$c_{ui} = S_0N(x_1)e^{-qT} - Ke^{-rT}N(x_1 - \sigma\sqrt{T}) - S_0e^{-qT}(H/S_0)^{2\lambda}[N(-y) - N(-y_1)] + Ke^{-rT}(H/S_0)^{2\lambda-2}[N(-y + \sigma\sqrt{T}) - N(-y_1 + \sigma\sqrt{T})]$$

However, in most cases, the price of the underlying asset S is monitored periodically and not continuously. In this case, the presented formulas should be adjusted to take into consideration the discrete nature of the monitoring. Suppose that the price is monitored m times. Then, the barrier level H should be replaced by $He^{0.5826\sigma\sqrt{T/m}}$ for an up-and-in or up-and-out-option and by $He^{-}0.5826\sigma\sqrt{T/m}$ for a down-and-in or down-and-out option.

In addition to simulating the price of the barrier option using Monte Carlo simulations, we present the results after applying the two variance-reduction methods described in the previous section, namely the antithetic variate and the control variate methods.

Figure 3 presents the results of the simulation experiments for an up-and-in call barrier option for different numbers of simulations. We observe that the estimated prices converge as the number of simulations increases, similar to the case of the Asian call option presented in the previous section. In this simulation study, we can observe that the control variate method exhibits the lowest variability among the three methods considered, outperforming the antithetic variable. As in the Asian option case, we average through the fixings and inherently reduce the variability of the price it is likely that the superior performance of the control variate estimator was not as apparent compared to antithetic variable. In this case, instead, we can observe a much more stable performance. Lastly, the estimations of the naive

Monte Carlo and the antithetic variable methods exhibit a similar convergence as the number of simulations increases, even if less than in the Asian Option case.

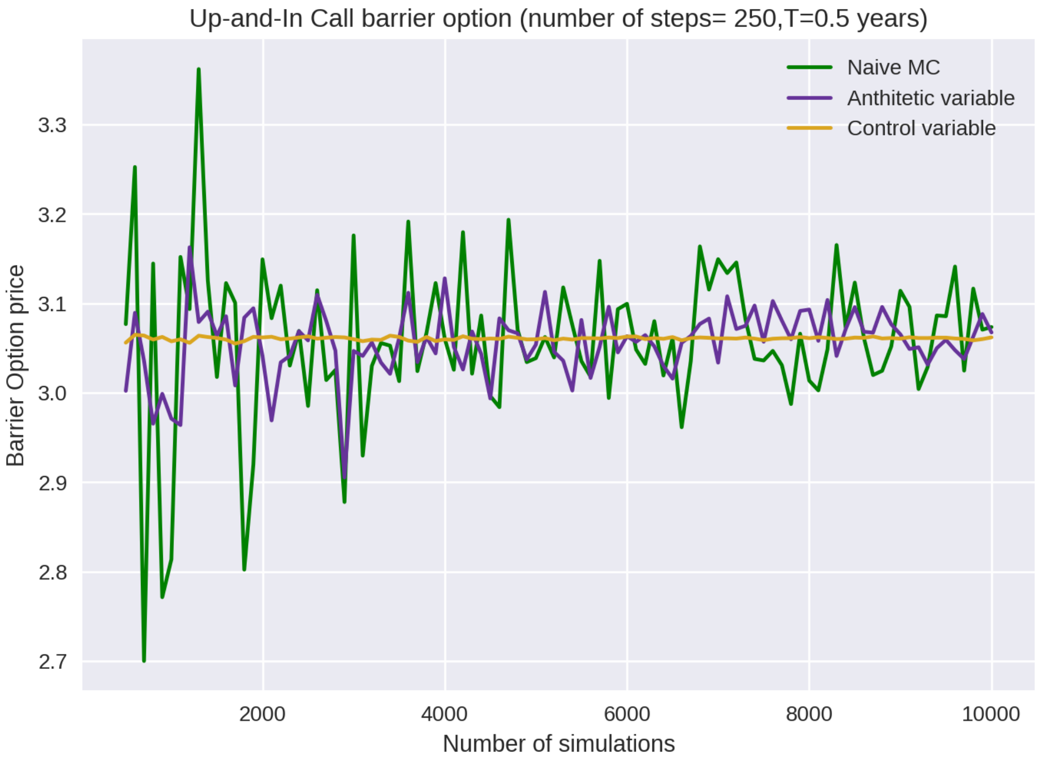

Figure 3: Convergence of Monte Carlo Estimator for a Up-and-In call barrier option

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